On the odderon intercept in perturbative QCD

N. Armesto and M. A. Braun*
Departamento de Física de Partículas,
Universidade de Santiago de Compostela,
15706–Santiago de Compostela, Spain

Abstract

Arguments are presented for the odderon intercept being exactly equal to unity. A variational method is presented based on a complete system of one-gluon functions. For the odderon, the highest intercept calculated by this method is $1-(3\alpha_s/\pi)\,0.45$.

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* Visiting Professor IBERDROLA. On leave of absence from the Department of High Energy Physics, University of St. Petersburg, 198904 St. Petersburg, Russia.

1 Introduction

Much attention has recently been devoted to the perturbative "hard", or BFKL, pomeron [1], especially in relation to the study of the small x behaviour of the deep inelastic scattering structure functions (see a recent review in [2]). In application to soft phenomena, the value of the pomeron intercept is of principal importance. For the BFKL pomeron it is considerably above unity:

$$\alpha_{BFKL}(0) = 1 - (3\alpha_s/\pi)E_0,$$

where the "energy" E_0 is equal to $-4\ln 2$, and α_s is the (fixed) QCD coupling constant [1]. As a result, to obtain a unitary amplitude one has to take into account more than two, in fact, any number of interacting reggeized gluons. This problem simplifies in the large N_c limit, when it reduces to summing all multipomeron exchanges [3].

For the negative signatured amplitude the lowest order contribution comes from the exchange of an odderon, a state formed from three reggeized gluons in a symmetric colour configuration [4]. Its relative importance is controlled by the odderon intercept. Should it also lie above unity, unitarization would require summing any number of exchanged odderons as well.

It has not been possible to obtain a complete solution of the odderon equation ("the BKP equation") up to now. Certain encouraging ideas have been proposed, however, in [5-7], based on the conformal symmetry of the equation and the Yang-Baxter technique. In [8] the conformal symmetry was used to reduce the problem to an one-dimensional equation. Variational calculations on the basis of this approach, with a relatively simple trial function containng two free parameters, gave an intercept above unity, although lower than for the pomeron [9]:

$$\alpha_{odd}(0) = 1 - (3\alpha_s/\pi)E_{odd},\ E_{odd} < -0.37.$$

A different scheme of variational calculations of the odderon intercept was adopted in our paper [10]. We studied the odderon energy matrix in a non conformally invariant basis of functions, whose number was taken rather large (up to 3368), which corresponds to a correspondingly large number of variational parameters. Our best result for the energy however turned out positive

$$E_{odd} < 0.45$$

corresponding to an intercept below unity. In view of the variational character of the calculations this result evidently does not contradict [9] but seems to be much weaker. For that reason we did not give much importance to our result at the time when it was obtained, so that it remained unpublished.

However, further study of the odderon problem has given us some motivation to believe that this result may be closer to reality than the one obtained in [9]. The point is that the equation for the odderon at rest admits a simple solution corresponding to the energy exactly equal to zero and the intercept $j = 1$. This solution is based on the so-called bootstrap relation in the BFKL theory [11] and we call it the bootstrap solution. One can view this solution as a true bound state (normalizable). It is nodeless and possesses a maximal possible symmetry. Therefore, from the experience gained in quantum-mechanical problems, we can expect it to correspond to the ground state of the odderon, that is, to the lowest energy possible.

In Sec. 2 we discuss this point in more detail giving some mathematical arguments in favour of this result. In view of this, our calculations aquire a better status, which gives us a reason to present them in this paper, in Secs. 3-5. Some conclusions are drawn in Sec. 6. All mathematical details are relegated to Appendices.

2 The bootstrap solution as the ground state of the odderon

As shown in [4] the odderon wave function in the transversal momentum space $\psi(q_1, q_2, q_3)$ satisfies a Schroedinger-like equation

$$H\psi = E\psi. \tag{1}$$

The hamiltonian H is a sum of kinetic terms and pair interactions between the three gluons:

$$H = T_1 + T_2 + T_3 + U_{12} + U_{23} + U_{31}. \tag{2}$$

Each kinetic term is given by the gluonic Regge trajectory with a minus sign in units $3\alpha_s/\pi$:

$$T_1 = T(q_1) = -\omega(q_1) = \frac{\eta(q_1)}{4\pi} \int \frac{d^2q_1'}{\eta(q_1')\eta(q_1 - q_1')}, \tag{3}$$

where, with an infrared regularization provided by the gluon mass m,

$$\eta(q) = q^2 + m^2. \tag{4}$$

The interaction terms U are integral operators in the momentum space of the three gluons with an integration measure

$$d\mu = \frac{d^2q_1 d^2q_2 d^2q_3 \delta^2(q_1 + q_2 + q_3 - q)}{\eta(q_1)\eta(q_2)\eta(q_3)} \tag{5}$$

where q is the (fixed) total transverse momentum of the odderon. In this section we shall consider only the case $q = 0$ so that $q_1 + q_2 + q_3 = 0$ will be assumed. Explicitly the kernel, say, of U_{12} is given by

$$U_{12}(q_1, q_2, q_3|q_1', q_2', q_3') = \eta(q_3)\delta^2(q_3 - q_3')V_{12}(q_1, q_2|q_1', q_2'), \tag{6}$$

where V_{12} is the BFKL interaction for two gluons in a vector colour state:

$$V_{12}(q_1,q_2|q_1',q_2') = -\frac{1}{4\pi}\left[\frac{\eta_(q_1)\eta_(q_2') + \eta_(q_1')\eta_(q_2)}{\eta(q_1-q_1')} - \eta(q_1+q_2)\right]. \tag{7}$$

(Note that this interaction is twice smaller than the one for the vacuum chanel which enters the standard pomeron equation).

It is well-known that like the pomeron equation, the BKP equation (1) is infrared stable, that is, it remains meaningful if one puts $m = 0$ in (4) [4]. However, for our purpose it will be convenient to proceed in a different manner. Rescaling $q \to q/m$ we eliminate m from Eq. (1) altogether, the resulting $\eta(q)$ having the form

$$\eta(q) = q^2 + 1. \tag{8}$$

Thus one observes that Eq. (1) (homogeneous!) is in fact independent of m and so are the energy eigenvalues E, which is of no wonder, since they are dimensionless. In the following we shall discuss Eq. (1) in the infrared regularized form provided by η given by (8). It clearly shows that the singularities of the equation come only from the ultraviolet region when $q_{1,2,3} \to \infty$.

The crucial point for our discussion is the existence of the bootstrap solution which is a constant:

$$\psi_B(q_1,q_2,q_3) = \psi_0. \tag{9}$$

As mentioned, the bootstrap identity is essential for this:

$$\int \frac{d^2q_1'}{\eta(q_1')\eta(q_2')} V_{12}(q_1,q_2|q_1',q_2') = \omega(q_1) + \omega(q_2) - \omega(q_1+q_2), \tag{10}$$

which is a consequence of the structure of V and ω in terms of η and valid for an arbitrary $\eta(q)$. Applying this identity we find

$$H\psi_B = \psi_0(-\omega(q_1) - \omega(q_2) - \omega(q_3)$$

$$+2\omega(q_1) + 2\omega(q_2) + 2\omega(q_3) - \omega(q_1+q_2) - \omega(q_2+q_3) - \omega(q_3+q_1)) = 0.$$

Here the first three terms come from the kinetic part and the rest from the interaction according to (10). This expression vanishes because at $q = 0$ we have $\omega(q_1+q_2) = \omega(q_3)$ etc. Evidently the solution (9) is normalizable with the measure given by (5) and η taken according to (8):

$$\|\psi_B\|^2 = |\psi_0|^2 \int d\mu < \infty. \tag{11}$$

Therefore it represents a true bound state of the three gluons with a zero energy. It evidently possesses the maximal symmetry possible. As mentioned, our experience with quantum-mechanical problems then suggests that it is the ground state of the system.

To somewhat strengthen this proposition, we take a bit more mathematical point of view. The integral equation (1) is singular for two reasons. One is evident and comes from the pairwise nature of the interaction. It is common to all quantum mechanical three-body problems. As is well-known, it can be cured by going over to the equivalent Faddeev matrix equations for parts of the wave function. However there is another source of the singularity related to a bad ultraviolet behaviour of the kinetic terms: they grow very slow with $q \to \infty$ (only logarithmically) making the equation badly singular. This singularity persists also in the corresponding two-body equation for the BFKL pomeron and is responsible for its spectrum to be different from the "free" equation with $V = 0$. To cure this singularity we introduce an ultraviolet cutoff into η substituting (8) by

$$\eta_\nu(q) = \eta(q) \exp(\nu q^2), \quad \nu > 0 \tag{12}$$

With $\nu > 0$ the kinetic terms grow exponentially with the momentum. One easily finds that

$$\omega_\nu(q)_{q\to\infty} \simeq -\frac{2}{\nu q^2} \exp(\frac{1}{2}\nu q^2). \tag{13}$$

Now one can apply the standard methods to study the spectrum of the odderon regularized in the ultraviolet.

Let us illustrate how it works with a much simpler two-body problem of the pomeron at rest. After changing its wave function $\psi(q_1, q_2)$ with $q_1 + q_2 = 0$ according to

$$\psi(q_1, q_2) = \eta_\nu(q_1)\phi(q_1)$$

the pomeron equation reduces to

$$(H_0(q) - E)\phi(q) = -\int d^2q' U(q, q')\phi(q'), \tag{14}$$

with $H_0(q) = -2\omega_\nu(q)$ and the kernel U given by

$$U(q, q') = -\frac{1}{2\pi}(\frac{2}{\eta_\nu(q - q')} - \frac{\eta_\nu(0)}{\eta_\nu(q)\eta_\nu(q')}) \tag{15}$$

From (14) we go over to an equivalent eigenvalue equation:

$$K(E)\chi_\lambda = \lambda(E)\chi_\lambda, \tag{16}$$

where

$$K(E) = (H_0 - E)^{-1/2} U (H_0 - E)^{-1/2}. \tag{17}$$

Evidently the energy eigenvalues E for Eq. (14) are determined from the ones for (16) by the equation

$$\lambda(E) = -1 \tag{18}$$

and the corresponding eigenfunctions are related by

$$\phi = (H_0 - E)^{-1/2}\chi_{-1}. \tag{19}$$

Eq. (16) has usually much better properties as compared to the initial Schroedinger equation (14). The denominators in (17) normally make the kernel K to be of the Fredholm type, provided that both H_0 and U are not too badly behaved and that E does not lie inside the spectrum of H_0. Then Eq. (16) possesses only a discrete spectrum of eigenvalues, which may be ordered according to their absolute values as

$$|\lambda_1| > |\lambda_2| > ....$$

With a finite norm of K, the number of λ's with an absolute value more than unity is finite. According to (18) this means that only a finite number of discrete eigenvalues E may exist below the continuum spectrum, the latter coinciding with that of H_0.

In the pomeron case we find by a direct calculation that with $\nu > 0$ the norm of the kernel $K(E)$ is finite for all E below the minimal value of H_0:

$$\|K_\nu(E)\|^2 = \int d^2q d^2q' |K_\nu(E;q,q')|^2 < \infty. \tag{20}$$

One can even calculate the norm at $E = 0$ and $\nu \to \infty$:

$$\|K_\nu(0)\|^2_{\nu\to\infty} = 64/9. \tag{21}$$

Note that this limit corresponds to $q^2 \sim 1/\nu$ and consequently to an $\eta(q)$ in the form of a pure exponential

$$\eta_\nu(q)_{\nu\to\infty} \simeq \exp(\nu q^2). \tag{22}$$

What do these results tell us about the physically interesting pomeron spectrum at $\nu \to 0$? Very little indeed. One observes that with a large ν there can be at most 7 eigenvalues λ_i with an absolute value greater than unity at $E = 0$. This means that there are at most 7 discrete negative energy levels at large ν. As ν diminishes the norm of K becomes larger and the number of discrete negative levels E also increases. When $\nu \to 0$ the norm blows up to infinity and so does the number of negative energy levels. The latter finally form a cut which completely fills the gap between the BFKL energy level $E_{BFKL} = -(12\alpha_s\pi)\ln 2$ and the kinetic energy threshold $E_0 = 3\alpha_s/2\pi$. Except for this behaviour with the change of ν, no new information can be extracted from this approach in the pomeron case.

However, applied to the odderon case this argument leads to certain important consequences. One notices that with η changed to η_ν the bootstrap identity (10) remains valid and thus the bootstrap solution persists with the energy eigenvalue $E = 0$ at any ν. This

means that for the odderon equation similar to (16) a curve $\lambda_B(E)$ always exists which passes through minus unity at $E = 0$: $\lambda_B(0) = -1$. Since different λ's cannot cross with changing ν and since $|\lambda|$'s are expected to fall monotonously with $|E|$ for E below the spectrum of H_0, we find that the curve $\lambda_B(E)$ divides all λ's into two separate sets: those which lie above it and those which lie below it (Fig. 1), and that this division is conserved as ν changes. Since we expect the norm of the operator K in the properly formulated (Faddeev) equation to be finite at $E = 0$, the number of λ's below λ_B has also to be finite and it is conserved with the changing ν. Then, as a first result, we find that the solutions of the physical odderon equation with $\nu = 0$ with negative energies form a discrete finite set. A second result is that the number of such solutions can be studied at any chosen value of ν, in particular at large ν, since this number is adiabatically conserved with ν. The odderon theory with a large ν and the exponential η of the form (22) is much simpler than for the physical odderon with $\nu = 0$. Direct variational estimates reveal that in this case there are no solutions with nonpositive energies E at large ν except the bootstrap one (see Appendix 1.). This brings us to the conclusion that the bootstrap solution is indeed the one with the lowest energy.

Of course, this argument is not absolutely rigorous. To make it such, one has to find the norm of the corresponding Faddeev kernel (a 3×3 skew- symmetric matrix of two-body scattering matrices). Before that one has to study these pair scattering matrices and show that their properties are no worse than those of the interactions U_{ik} given by (6) and (7). This seems realizable although rather difficult. Also, even knowing that the norm of K is finite, one cannot exclude in principle some bizarre behaviour of the eigenvalues λ which might invalidate the above logic. For that reason we do not consider our derivation as a definite proof but rather as a strong argument in favour of the bootstrap solution to represent the odderon ground state.

3 Variational calculations of the odderon ground state energy

To perform variational estimates of the odderon ground state energy it is more convenient to set $m = 0$ from the start, to simplify the explicit form of the Hamiltonian. To make it non-singular we pass to a new wave function by a substitution

$$\psi \to \prod_{i=1}^{3} q_i^2 \psi. \tag{23}$$

We shall not fix the total momentum of the odderon. Then for the new ψ and with $m = 0$ the metric (5) changes to

$$d\mu = \prod_{i=1}^{3} q_i^2 d^2 q_i. \tag{24}$$

The odderon equation (1) becomes

$$H\psi = E\prod_{i=1}^{3} q_i^2\psi, \tag{25}$$

The Hamiltonian can be written as [8]

$$H = (1/2)(H_{12} + H_{23} + H_{31}) \tag{26}$$

where H_{ik} is the BFKL Hamiltonian for gluons i and k in units $3\alpha_s/2\pi$, which in the limit $m = 0$ acts on the wave function as

$$H_{ik}\psi = \prod_{j=1}^{3} q_j^2(\ln q_i^2 q_k^2 + 4\mathbf{C})\psi + \prod_{j=1, j\neq i,k}^{3} q_j^2 \left[q_i^2 \ln(r_{ik}^2/4)\, q_k^2 + (i \leftrightarrow k)\right]\psi + 2(q_i+q_k)^2\psi(r_{ik}=0), \tag{27}$$

where $r_{ik} = r_i - r_k$ is the (transversal) distance between the gluons and $\mathbf{C}$ is the Euler constant.

The solution of (25) may be found by a variational approach, searching the minimum value of the functional

$$\Phi = \int \prod d^2 q_i \psi^* H\psi, \tag{28}$$

with the normalization condition

$$\int \prod d^2 q_i \psi^* \prod_{i=1}^{3} q_i^2\psi = 1. \tag{29}$$

Using the symmetry of the wave functions in the three gluons, one can study a simpler functional

$$\mathcal{E} = (1/2)\int \prod_{i=1}^{3} d^2 q_i \psi^* H_{12}\psi. \tag{30}$$

The odderon energy is determined by the minimal value ϵ_3 of $\mathcal{E}$ according to

$$E_{odd} = (3/2)\epsilon_3. \tag{31}$$

Note that the minimal value ϵ_2 of the same functional on functions $\psi(q_1, q_2)$ for two gluons determine the pomeron energy: $E_0 = \epsilon_2$.

We expand the wave function in a sum of products of individual gluon functions:

$$\psi(r_1, r_2, r_3) = \sum_{\alpha_1,\alpha_2,\alpha_3} c_{\alpha_1,\alpha_2,\alpha_3} \prod_{i=1}^{3} \psi_{\alpha_i}(r_i), \tag{32}$$

where the one-gluon functions $\psi_\alpha(r_i)$ form a discrete complete set and are orthonormalized according to (29):

$$\int d^2 r \psi_\alpha^* q^2 \psi_{\alpha'} = \delta_{\alpha,\alpha'}. \tag{33}$$

The coefficients $c_{\alpha_1,\alpha_2,\alpha_3}$ have to be symmetric in all α's by the requirement of the Bose symmetry and normalized according to

$$\sum_{\alpha_1,\alpha_2,\alpha_3} |c_{\alpha_1,\alpha_2,\alpha_3}|^2 = 1. \tag{34}$$

The two-gluon Hamiltonian H_{12} acts nontrivially only on the wave functions for the gluons 1 and 2. So the energy functional becomes

$$\mathcal{E} = \sum_{\alpha_1,\alpha_2,\alpha_1',\alpha_2',\alpha_3} c^*_{\alpha_1,\alpha_2,\alpha_3} c_{\alpha_1',\alpha_2',\alpha_3} \mathcal{E}_{\alpha_1,\alpha_2,\alpha_1',\alpha_2'}, \tag{35}$$

where the matrix $\mathcal{E}_{\alpha_1,\alpha_2,\alpha_1',\alpha_2'}$ is the two-gluon energy in the basis formed by functions ψ_α. With this matrix known, the problem of minimization of the functional $\mathcal{E}$ reduces to finding the minimal value of a cuadratic form, that is, the minimal eigenvalue of the matrix $\mathcal{E}_{\alpha_1,\alpha_2,\alpha_1',\alpha_2'}$ considered as a matrix in independent initial and final 3-gluon states. The latter means that this matrix should be multiplied by unity matrix for the third gluon and then symmetrized in all initial and final gluons. The procedure is quite straightforward, once the basic functions ψ_α are chosen. It however involves a numerical evaluation of the energy matrix elements and a diagonalization of the matrix, whose dimension is rapidly growing with the number of the basic functions taken into account.

4 Two-gluon energy matrix for given angular momenta

Introducing the individual gluon angular momentum l we take ϕ trivially:

$$\psi_\alpha(\mathbf{r}) = \psi_{k,l}(r) \exp il\phi, \tag{36}$$

where $k = 0, 1, 2, ...$ enumerates the radial functions. In the following, instead of r, we shall use the variable $z = \ln r^2$ in most cases. In terms of z and ϕ

$$q^2 \psi_{k,l}(\mathbf{r}) = -(4/r^2)(\partial_z^2 - (1/4)l^2)\psi_{k,l}(\mathbf{r}). \tag{37}$$

Wave functions with different values of the angular momentum are automatically orthogonal. For coinciding l the normalization condition for the radial functions reduces to the standard form for functions

$$\xi_{k,l}(z) = (\partial + |l|)\psi_{k,l}(z), \tag{38}$$

which should satisfy

$$\int dz \xi^*_{k,l}(z)\xi_{k',l}(z) = (1/4\pi)\delta_{kk'}. \tag{39}$$

We assume that the radial functions are chosen to be real.

With the angular dependence of the wave function explicitly given by (36), one can do the azymuthal integrals in the potential energy in a straightforward manner. Let $\alpha_i = \{k_i, l_i\}$ and take the transition between two gluon states $\alpha_1, \alpha_2 \to \alpha_3, \alpha_4$. Evidently the total angular momentum is conserved so that the energy matrix elements are zero unless $l_1 + l_2 = l_3 + l_4$. According to (27) the potential energy consists of two parts, the first part U given by an essentially Coulomb interaction and the second one Q given by a contact interaction, proportional to their total momentum squared. Let us begin with the Coulomb part U. Its two terms evidently give the same contribution due to the symmetry under the interchange of gluons 1 and 2. So we can take only one of them and drop the factor 1/2. Denote

$$\eta_{k,l}(z) = (\partial^2 - (1/4)l^2)\psi_{k,l}(z). \tag{40}$$

Then after doing the azymuthal integration we obtain

$$U_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 16\pi^2 \int dz_1 dz_2 \eta_{\alpha_1}(z_1)\psi_{\alpha_3}(z_1)\psi_{\alpha_2}(z_2)\eta_{\alpha_4}(z_2)U_l(z_1, z_2), \tag{41}$$

where $l = |l_1 - l_3| = |l_2 - l_4|$ is the angular momentum transfer and the function $U_l(z_1, z_2)$ is given by

$$U_l = -(1/l)\exp(-(l/2)|z_1 - z_2|), \quad l \neq 0, \tag{42}$$

and

$$U_0 = \max\{z_1, z_2\}. \tag{43}$$

The contact part Q involves gluonic wave functions taken at the same point. After performing the azymuthal integration and integrating once by parts in the variable z we obtain

$$Q_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 8\pi^2 \int dz \left[(\partial + (1/2)|l_1 + l_2|)\psi_{\alpha_1}\psi_{\alpha_2}\right]\left[(\partial + (1/2)|l_3 + l_4|)\psi_{\alpha_3}\psi_{\alpha_4}\right]. \tag{44}$$

One can somewhat simplify this expression by noting that

$$(\partial + (1/2)|l_1 + l_2|)\psi_{\alpha_1}\psi_{\alpha_2} = \xi_{\alpha_1}\psi_{\alpha_2} + \psi_{\alpha_1}\xi_{\alpha_2} + \Delta_{12}\psi_{\alpha_1}\psi_{\alpha_2}, \tag{45}$$

where $2\Delta_{12} = |l_1 + l_2| - |l_1| - |l_2|$ and similarly for the second factor in (44). Then finally

$$Q_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 8\pi^2 \int dz(\xi_{\alpha_1}\psi_{\alpha_2} + \psi_{\alpha_1}\xi_{\alpha_2} + \Delta_{12}\psi_{\alpha_1}\psi_{\alpha_2})(\xi_{\alpha_3}\psi_{\alpha_4} + \psi_{\alpha_3}\xi_{\alpha_4} + \Delta_{34}\psi_{\alpha_3}\psi_{\alpha_4}). \tag{46}$$

The kinetic energy is easier to calculate in the momentum space. So we transform the basic functions to the momentum space according to

$$\psi_\alpha(\mathbf{q}) = \int (d^2r/2\pi)\psi_\alpha(\mathbf{r})\exp(-i\mathbf{qr}) = (-i)^l \exp il\phi \int r dr \psi_{k,l}(z)J_l(qr). \tag{47}$$

where J_l is the Bessel function. To do the integral over r it is convenient to introduce a Fourier transform of the function ψ with respect to the variable z:

$$\psi_{k,l}(z) = \int (d\nu/\sqrt{2\pi})\phi_{k,l}(\nu) \exp i\nu z. \tag{48}$$

Putting this representation in (47) and doing the r-integration we obtain

$$\psi_{k,l}(\mathbf{q}) = (2/q^2) \exp il\phi \int (d\nu/\sqrt{2\pi}) f_{k,l}(\nu) q^{-2i\nu}, \tag{49}$$

with

$$f_{k,l}(\nu) = (-i)^{|l|} 2^{2i\nu} (|l|/2 + i\nu)\phi_{k,l}(\nu)\Gamma(|l|/2 + i\nu)/\Gamma(|l|/2 - i\nu). \tag{50}$$

With the gluon wave functions in the momentum space given by (49), both radial and azymuthal integration in $\mathbf{q}$ are easily done. The final matrix element of the kinetic energy T results as

$$T_{\alpha_1,\alpha_3} = -4\pi i \int d\nu f^*_{k_1,l_1}(\nu)(\partial/\partial\nu) f_{k_3,l_1}(\nu), \quad l_1 = l_3 \tag{51}$$

The differentiation gives

$$(\partial/\partial\nu) f_{k_3,l_1}(\nu) = f_{k_3,l_1}(\nu) \left[2i \ln 2 + 2i \mathrm{Re}\, \psi(|l_1|/2 + i\nu) + (\partial/\partial\nu) \ln \left[(|l|/2 + i\nu)\phi_{k,l}(\nu)\right]\right]. \tag{52}$$

Correspondingly the kinetic energy matrix element separates into terms

$$T^{(1)}_{\alpha_1,\alpha_3} = 8\pi \int d\nu f^*_{k_1,l_1}(\nu) f_{k_3,l_1}(\nu) \left[\ln 2 + \mathrm{Re}\, \psi(|l_1|/2 + i\nu)\right] \tag{53}$$

and

$$T^{(2)}_{\alpha_1,\alpha_3} = -4\pi i \int d\nu \left[(|l_1|/2 + i\nu)\phi_{k_1,l_1}(\nu)\right]^* (\partial/\partial\nu) \left[(|l_1|/2 + i\nu)\phi_{k_3,l_1}(\nu)\right]. \tag{54}$$

The function $(|l|/2 + i\nu)\phi_{k,l}(\nu)$ is nothing but the Fourier transform of $\xi_{k,l}(z)$ with respect to z. Correspondingly we denote it as

$$(|l|/2 + i\nu)\phi_{k,l}(\nu) \equiv \xi_{k,l}(\nu). \tag{55}$$

The part $T^{(2)}$ can then be written as

$$T^{(2)}_{\alpha_1,\alpha_3} = -4\pi i \int d\nu \xi_{k_1,l_1}(\nu)^* (\partial/\partial\nu) \xi_{k_3,l_1}(\nu). \tag{56}$$

The orthonormalization property (39) transforms into the analogous property in the ν space

$$\int d\nu \xi^*_{k,l}(\nu)\xi_{k',l}(\nu) = (1/4\pi)\delta_{kk'}. \tag{57}$$

Noting that $f^*_{k,l}(\nu) f_{k',l}(\nu) = \xi^*_{k,l}(\nu)\xi_{k',l}(\nu)$ we observe that the term $\ln 2$ in (53) will add a constant $2 \ln 2$ to the energy. Separating another constant term $2\psi(1)$ we finally present the part $T^{(1)}$ in the final form

$$T^{(1)}_{\alpha_1,\alpha_3} = 2(\ln 2 + \psi(1))\delta_{\alpha_1,\alpha_3} + 8\pi \int d\nu \xi^*_{\alpha_1}(\nu)\xi_{\alpha_3}(\nu) \left[\mathrm{Re}\, \psi(|l_1|/2 + i\nu) - \psi(1)\right]. \tag{58}$$

The first, constant, term cancels an identical one in the initial Hamiltonian (27). Using the representation

$$\psi(x) - \psi(1) = \int_0^\infty dt \left[\exp(-t) - \exp(-xt)\right] / (1 - \exp(-t)) \tag{59}$$

and the othornormalization property of the set ξ_α we may cast $T^{(1)}$ in the form

$$T^{(1)}_{\alpha_1,\alpha_3} = 2 \int_0^\infty (dt/(\exp t - 1))(\delta_{\alpha_1\alpha_3} - \exp(t(1 - |l_1|/2)) g_{\alpha_1\alpha_3}(t)), \tag{60}$$

where

$$g_{\alpha_1\alpha_3}(t) = 4\pi \int d\nu \xi^*_{\alpha_1}(\nu)\xi_{\alpha_3}(\nu) \cos \nu t. \tag{61}$$

Note that (59) is not valid for $\mathrm{R}e\, x = 0$. Therefore this formula cannot be applied when the gluon orbital momentum is zero. In this case one may use

$$\psi(i\nu) + \psi(-i\nu) = \psi(1 + i\nu) + \psi(1 - i\nu),$$

which formally corresponds to changing the angular momentum to be equal to 2.

As to the second part of the kinetic energy $T^{(2)}$, it turns out to be cancelled by a similar contribution coming from the monopole part of the Coulomb interaction for the angular momentum transfer equal to zero (see Appendix 2.).

Most of the contributions to the energy presented in this section can hardly be further simplified and were used in the numerical calculations as they stand. The exception is the monopole part of the Coulomb interaction corresponding to (41) with $l = 0$ (Eq. (43)). This part contains contributions which cancel the term $T^{(2)}$ in the kinetic energy and partially the contact interaction contribution for $l = 0$. The cancellation between the monopole Coulomb interaction and the kinetic term $T^{(2)}$ is responsible for the scale invariance of the energy. Calculation of the monopole Coulomb part is discussed in Appendix 2.

5 The choice of basic functions and the numerical results

A natural orthonormal discrete basis for $z = \ln r^2$ varying from $-\infty$ to $+\infty$ is formed by the harmonic oscillator proper functions. Thus we choose functions $\xi_{k,l}(z)$ independent of l and given by

$$\xi_k(z) = c_k H_k(z) \exp(-z^2/2), \tag{62}$$

where H_k are the Hermite polinomals and c_k are determined by the normalization condition (39) to be

$$c_k^2 = 1/(4\pi^{3/2} 2^k k!). \tag{63}$$

The Fourier transformation to the ν space gives

$$\xi_k(\nu) = (-i)^k c_k H_k(\nu) \exp(-\nu^2/2). \tag{64}$$

In the coordinate space the function $\eta_{k,l}(z)$ is obtained from ξ by differentiation:

$$\eta_{k,l}(z) = (\partial - (1/2)|l|)\xi_k(z). \tag{65}$$

The function $\psi_{k,l}$ is obtained from ξ_k as a solution of the differential equation

$$\xi_k(z) = (\partial + (1/2)|l|)\psi_{k,l}(z), \tag{66}$$

with a boundary condition $\psi_{k,l}(-\infty) = 0$. It is given by an integral

$$\psi_{k,l}(z) = \int_{-\infty}^{z} dz' \xi_k(z') \exp(-|l|(z-z')/2). \tag{67}$$

With this set of functions the potential part of the energy was calculated numerically. As to the kinetic part, the function g entering (60) can be found analytically. For transition $k, l \to k', l$ it is equal to zero if $k + k'$ is odd. For even $k + k' = 2s$ and $k \geq k'$

$$g_{kk'}(t) = 4\pi^{3/2}(-1)^d c_k c_{k'} \exp(-t^2/4) \sum_{p=0}^{k'} 2^p p! C_k^p C_{k'}^p (-t^2)^{s-p}, \tag{68}$$

where $2d = k - k'$.

After the energy matrix $\mathcal{E}_{\alpha_1,\alpha_2,\alpha_1',\alpha_2'}$ is calculated and properly symmetrized in the three gluons, its lowest eigenvalue is determined, which gives an upper limit on the exact odderon energy according to Eq. (31). To study the minimal energy only states with the total angular momentum equal to zero have been included.

The selected set of basic one-gluon functions is characterized by the maximal value of the angular momentum included l_{max} and numbers of radial functions included for each wave. As calculations show, best results are obtained when one raises l_{max} and the number of radials in all waves simultaneously. So we present here the results for the case when the number of radials r is the same for all angular momenta and is equal to the number of angular momenta included: $r = l_{max} + 1$. Such a set of functions is thus characterized by a single parameter r. With a growth of r the number of states N rises very rapidly. In our calculations the number r was limited to 6.

To study convergence of the procedure, it was first applied to the pomeron, with only two gluons, where the exact energy is known. The maximal $r = 6$ corresponds to 201 basic two-gluon states in this case. For the odderon with $r = 6$ the number of basic states rises to 3368.

The results of the calculations of the ground state energies of the BFKL pomeron ϵ_2 and the odderon ϵ_3 are presented in the Table for different values of r. One observes that for the pomeron the obtained energies are still rather far from the exact value. Thus the convergence of the method is rather slow. The Table also reveals that the odderon energy is essentially

larger than the pomeron one for a given r. So our results confirm that, in all probability, the odderon intercept is lower than that of the BFKL pomeron. Our values for the odderon energy stay positive, approaching zero quite slowly. As discussed in Sec. 2, this seems to favour the minimal odderon energy being equal to zero.

6 Conclusions

We have presented arguments that the odderon intercept is exactly equal to unity, starting from the explicitly known bootstrap solution. This solution corresponds to the odderon at rest. However for lack of scale, the intercept should be the same for nonzero momentum transfers. Our variational calculations seem to confirm this result.

If it is correct then experimentally we would expect the difference between the cross-sections on a particle and its antiparticle to stay constant at high energies. Also the odderon effects in the cross-sections themselves should decrease with energy, since the pomeron contribution rises with it.

On the theoretical side, the intercept of the odderon being equal to one means that exchanges of more than one odderon are of no importance. Then the unitarization program essentially reduces to summing only pomerons. As mentioned this can be realized in the high-colour limit. However to do it for the negative signatured amplitude, one has to know how to couple the odderon to an external sourse perturbatively. This seems to be an unsolved problem.

7 Acknoledgements

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8 Appendix 1. Variational estimates of the odderon ground state energy with an exponential η

With (22) the form of the kinetic part and the interaction, as well as that of the measure (5), simplify drastically and reduce to Gaussians. Then one can calculate the average energy quite easily, provided the trial function is also taken Gaussian or a sum of Gaussians. We performed variational estimates of the ground state energy choosing the trial function in the form

$$\psi(q_1,q_2,q_3)=\sum_i c_i e^{-\beta_i(q_1^2+q_2^2+q_3^2)},\quad q_1+q_2+q_3=0, \tag{69}$$

with variational parameters c_i and $\beta_i \geq 0$.

The norm of ψ turns out to be

$$\|\psi\|^2 = \sum_{ik} c_i c_k N_{ik}, \tag{70}$$

with

$$N_{ik} = (\pi^2/3)(\nu + \beta_{ik})^{-2}, \tag{71}$$

where we have defined $\beta_{ik} = \beta_i + \beta_k$.

The average energy in the state ψ results

$$\langle H \rangle = \sum_{ik} c_i c_k E_{ik}. \tag{72}$$

The matrix E_{ik} is a sum of three terms:

$$E_{ik} = \sum_{j=1}^{3} E_{ik}^{(j)}, \tag{73}$$

corresponding to the kinetic part ($j = 1$), the first term in the interaction (7) ($j = 2$) and to its second term ($j = 3$). They are given by

$$E_{ik}^{(1)} = (3\pi^2/8)\nu^{-1}(\nu + \beta_{ik})^{-1}(2\nu + 3\beta_{ik})^{-1}, \tag{74}$$

$$E_{ik}^{(2)} = -(3\pi^2/4)(\nu + \beta_{ik})^{-1}(2\nu^2 + 3\nu\beta_{ik} + 3\beta_i\beta_k)^{-1}, \tag{75}$$

$$E_{ik}^{(3)} = (3\pi^2/8)(\nu + \beta_i)^{-1}(\nu + \beta_k)^{-1}(2\nu + 3\beta_{ik})^{-1}. \tag{76}$$

Numerical calculations show that whatever the number of gaussians is taken in (69) and however the values of β_i are chosen, the matrix E_{ik} has no nonpositive eigenvalue unless all $\beta_i = 0$, which case corresponds to the bootstrap solution. Thus the bootstrap solution is the only one with nonpositive energy in the limit $\nu \to \infty$.

9 Appendix 2. Monopole part of the Coulomb interaction

Explicitly the monopole term contribution is given by

$$U_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 16\pi^2 \int_{-\infty}^{\infty} dz_1 \eta_{\alpha_1}(z_1)\psi_{\alpha_3}(z_1) z_1 \int_{-\infty}^{z_1} \psi_{\alpha_2}(z_2)\eta_{\alpha_4}(z_2) + (\alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_3). \tag{77}$$

Here and in the following it is assumed that $l = 0$, that is, $l_1 = l_3$ and $l_2 = l_4$. Introduce a function

$$\chi_{\alpha_2,\alpha_4}(z) = \int_{-\infty}^{z} dz' \psi_{\alpha_2}(z')\eta_{\alpha_4}(z'). \tag{78}$$

Once integrating by parts we find

$$\chi_{\alpha_2,\alpha_4}(z) = \psi_{\alpha_2}(z)\xi_{\alpha_4}(z) - \xi_{\alpha_2,\alpha_4}(z), \tag{79}$$

where the function $\xi_{\alpha_2,\alpha_4}(z)$ with two indices, symmetric in these, is defined as

$$\xi_{\alpha_2,\alpha_4}(z) = \int_{-\infty}^{z} dz' \xi_{\alpha_2}(z')\xi_{\alpha_4}(z'). \tag{80}$$

As $z \to \infty$, according to (39), $\xi_{\alpha_2,\alpha_4}(z) \to (1/4\pi)\delta_{\alpha_2,\alpha_4}$, so that

$$\chi_{\alpha_2,\alpha_4}(\infty) = -(1/4\pi)\delta_{\alpha_2,\alpha_4}.$$

Having this in mind we can rewrite (77) in the form

$$U_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 16\pi^2 \int_{-\infty}^{\infty} dz(\chi_{\alpha_3,\alpha_1}(z) + (1/4\pi)\delta_{\alpha_1,\alpha_3})' z\chi_{\alpha_2,\alpha_4}(z) + (\alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_3). \tag{81}$$

Integrating by parts, the integral transforms into

$$-16\pi^2 \int_{-\infty}^{\infty} dz(\chi_{\alpha_3,\alpha_1}(z) + (1/4\pi)\delta_{\alpha_1,\alpha_3})(z\chi'_{\alpha_2,\alpha_4}(z) + \chi_{\alpha_2,\alpha_4}). \tag{82}$$

The term coming from the product $\chi_{\alpha_3,\alpha_1} z\chi'_{\alpha_2,\alpha_4}$ cancels the contribution $(\alpha_1 \leftrightarrow \alpha_4, \alpha_2 \leftrightarrow \alpha_3)$ in (81) so that the monopole contribution becomes

$$U_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = -16\pi^2 \int_{-\infty}^{\infty} dz(\chi_{\alpha_3,\alpha_1}(z)\chi_{\alpha_2,\alpha_4}(z) + (1/4\pi)\delta_{\alpha_1,\alpha_3}(z\chi'_{\alpha_2,\alpha_4}(z) + \chi_{\alpha_2,\alpha_4}(z))). \tag{83}$$

Now we substitute the functions χ in (83) by the symmetric functions ξ using relation (79). Take the the first term in the integrand of (83). With (79) we obtain for it

$$\chi_{\alpha_3,\alpha_1}\chi_{\alpha_2,\alpha_4} = \psi_{\alpha_3}\xi_{\alpha_1}\psi_{\alpha_2}\xi_{\alpha_4} - \psi_{\alpha_3}\xi_{\alpha_1}\xi_{\alpha_2,\alpha_4} - \xi_{\alpha_3,\alpha_1}\psi_{\alpha_2}\xi_{\alpha_4} + \xi_{\alpha_3,\alpha_1}\xi_{\alpha_2,\alpha_4}.$$

Having in mind the subsequent symmetrization with respect to the interchange of gluons 1 and 2, we can change $\alpha_1 \leftrightarrow \alpha_2$ and $\alpha_3 \leftrightarrow \alpha_4$ in the second term. Summed with the third term it then gives

$$-\xi_{\alpha_3,\alpha_1}(\psi_{\alpha_2}\xi_{\alpha_4} + \xi_{\alpha_2}\psi_{\alpha_4}). \tag{84}$$

Recall now that $\xi_{\alpha_2} = (\partial + (1/2)|l_2|)\psi_{\alpha_2}$ and similarly for ξ_{α_4}. Integration by parts allows to substitute (84) by

$$(\xi_{\alpha_3}\xi_{\alpha_1} - |l_2|\xi_{\alpha_3,\alpha_1})\psi_{\alpha_2}\psi_{\alpha_4}. \tag{85}$$

So finally the first term in (83) leads to the following three contributions to the monopole Coulomb energy:

$$\tilde{U}^{(1)}_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = -16\pi^2 \int_{-\infty}^{\infty} dz\xi_{\alpha_3,\alpha_1}\xi_{\alpha_2,\alpha_4}, \tag{86}$$

$$U^{(2)}_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 16\pi^2|l_2| \int_{-\infty}^{\infty} dz\xi_{\alpha_3,\alpha_1}\psi_{\alpha_2}\psi_{\alpha_4} \tag{87}$$

and

$$U^{(3)}_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = -16\pi^2 \int_{-\infty}^{\infty} dz(\psi_{\alpha_3}\xi_{\alpha_1}\psi_{\alpha_2}\xi_{\alpha_4} + \xi_{\alpha_3}\xi_{\alpha_1}\psi_{\alpha_2}\psi_{\alpha_4}). \tag{88}$$

Of these terms the first is divergent in its present form. It will receive its meaning after adding new contributions coming from the rest of the terms in (83). For that reason we have denoted it with a tilda.

Now for the rest of the terms in (83). Changing the function χ by ξ according to (79) we have under the integral

$$\xi_{\alpha_2,\alpha_4} + z\xi'_{\alpha_2,\alpha_4} = \psi_{\alpha_2}\xi_{\alpha_4} - \xi_{\alpha_2,\alpha_4} + z\psi_{\alpha_2}(\partial - (1/2)|l_2|)\xi_{\alpha_4}.$$

Integration by parts transforms it into

$$-\xi_{\alpha_2,\alpha_4} - z\xi_{\alpha_2}\xi_{\alpha_4}. \tag{89}$$

The first term can be combined with (86) to give the final $U^{(1)}$:

$$U^{(1)}_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 16\pi^2 \int_{-\infty}^{\infty} dz \xi_{\alpha_2,\alpha_4}((1/4\pi)\delta_{\alpha_1,\alpha_3} - \xi_{\alpha_3,\alpha_1}). \tag{90}$$

Now the integral converges due to the property (39). Putting here the explicit form of the functions ξ_{α_i,α_k} and integrating over z we obtain the term $U^{(1)}$ in its definitive form:

$$U^{(1)}_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 16\pi^2 \int dz_1 dz_2 (z_1 - z_2)\theta(z_1 - z_2)\xi_{\alpha_1}(z_1)\xi_{\alpha_3}(z_1)\xi_{\alpha_2}(z_2)\xi_{\alpha_4}(z_4). \tag{91}$$

The second term in (89) gives the last contribution to the monopole energy

$$U^{(4)}_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 4\pi\delta_{\alpha_1,\alpha_3} \int_{-\infty}^{\infty} dz z \xi_{\alpha_2}\xi_{\alpha_4}. \tag{92}$$

This term cancels with the contribution $T^{(2)}$ to the kinetic energy. Indeed after the Fourier transformation to the ν space according to (48), the factor z goes into $i\partial/\partial\nu$. One can then see that (92) gives exactly the contribution $T^{(2)}$, Eq. (56), with an opposite sign and with gluons 1 and 2 interchanged, which is of no importance because of the subsequent symmetrization.

The term $U^{(3)}$ cancels with the part of the contact interaction Q, Eq. (46), which does not contain factors Δ:

$$Q^{(2)}_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 8\pi^2 \int_{-\infty}^{\infty} dz(\psi_{\alpha_3}\xi_{\alpha_1}\psi_{\alpha_2}\xi_{\alpha_4} + \xi_{\alpha_3}\xi_{\alpha_1}\psi_{\alpha_2}\psi_{\alpha_4} + \psi_{\alpha_3}\psi_{\alpha_1}\xi_{\alpha_2}\xi_{\alpha_4} + \xi_{\alpha_3}\psi_{\alpha_1}\xi_{\alpha_2}\psi_{\alpha_4}). \tag{93}$$

Summed with $U^{(3)}$ this part gives

$$(Q^{(2)} + U^{(3)})_{\alpha_1,\alpha_2;\alpha_3,\alpha_4} = 8\pi^2 \int_{-\infty}^{\infty} dz(\psi_{\alpha_1}\xi_{\alpha_2} - \xi_{\alpha_1}\psi_{\alpha_2})(\xi_{\alpha_3}\psi_{\alpha_4} + \psi_{\alpha_3}\xi_{\alpha_4}). \tag{94}$$

This expression is antisymmetric under the interchange of the gluons 1 and 2 and does not give any contribution to the energy.

So finally the only contributions which remain in the interaction for zero angular momentum transfer are $U^{(1)}$, $U^{(2)}$ and the part $Q^{(1)}$ of the contact interaction (46) which contains factors Δ.

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Table

Calculated values of the ground state energy per gluon multiplied by 2 (Eq. (31)) for the pomeron (ϵ_2) and odderon (ϵ_3) with different numbers r of radial functions and angular momenta included.

r	ϵ_2	ϵ_3
1	0.968	0.968
2	0.022	0.605
3	-0.475	0.454
4	-0.743	0.379
5	-0.912	0.331
6	-1.032	0.298
∞	-2.773	

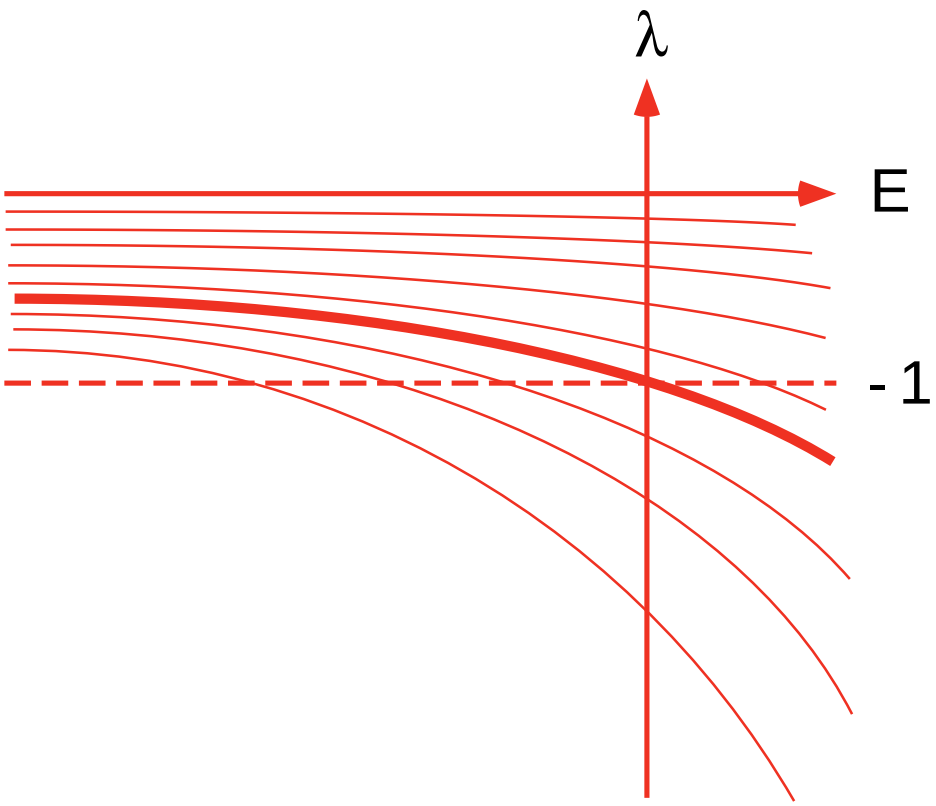

Fig. 1